Corrigendum: Local Projection Inference is Simpler and More Robust Than You Think*

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February 25, 2022

It has recently come to our attention that the high-level Assumption 3 on p. 1805 of Montiel Olea and Plagborg-Møller (2021) (henceforth "MOPM") is more restrictive than intended. As stated, the assumption allows for any VAR(1) process, both stationary and nonstationary, as well as VAR(p) processes with roots bounded away from the nonstationary part of the parameter space. However, several nonstationary VAR(p) models with $p > 1$ are ruled out.

In this note, we therefore propose a modification of Assumption 3 that can be verified for a wide range of VAR(p) models whose autoregressive parameters are contained in the parameter space defined on p. 1804 in MOPM. Our modified assumption is similar to Assumption 3 in the recent paper by Xu (2022), who applies the appropriate "Dickey-Fuller" transformation to the regressors.

If our modified Assumption 3 replaces the one in MOPM, all theoretical conclusions in our paper go through as originally stated. All econometric procedures, simulation results, efficiency calculations, and verbal discussions in our original paper are unaffected by the modification of Assumption 3.

Modified assumption

We first state and discuss the modified assumption, and then we indicate the corresponding minor changes to the proofs.

Our modification simply amounts to redefining the $np \times np$ matrix $G(A, h, \epsilon)$ introduced

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on p. 1805 in MOPM. In place of the old definition, consider

$$G(A,h,\epsilon) = \begin{pmatrix} I_n & 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \\ I_n & -\operatorname{diag}(\tilde\rho(A,\epsilon)) & 0_{n\times n} & 0_{n\times n} & \dots & 0_{n\times n} \\ 0_{n\times n} & I_n & -\operatorname{diag}(\tilde\rho(A,\epsilon)) & 0_{n\times n} & \dots & 0_{n\times n} \\ & & \ddots & \ddots & & \\ 0_{n\times n} & \dots & \dots & 0_{n\times n} & I_n & -\operatorname{diag}(\tilde\rho(A,\epsilon)) \end{pmatrix}^{-1}$$
$$\times \begin{pmatrix} \operatorname{diag}(g(\rho_1^*(A,\epsilon),h),\dots,g(\rho_n^*(A,\epsilon),h)) & 0_{n\times n(p-1)} \\ 0_{n(p-1)\times n} & I_{n(p-1)} \end{pmatrix},$$

where both matrices above are $np \times np$, and we define $\tilde\rho(A,\epsilon) \equiv (\tilde\rho_1(A,\epsilon),\dots,\tilde\rho_n(A,\epsilon))'$ and $\tilde\rho_i(A,\epsilon) \equiv \operatorname{sign}(\rho_i(A))\max\{|\rho_i(A)|,\epsilon\}$ for $i = 1,\dots,n$.[1] Here and in the following we use several objects defined on p. 1805 in MOPM; in particular, $g(\rho,h)^2 \equiv \min\{\frac{1}{1-|\rho|},h\}$, $\rho_i^*(A,\epsilon) \equiv \max\{|\rho_i(A)|, 1-\epsilon/2\}$, and $\rho_i(A)$ is the i-th potentially near-unity root for any VAR coefficients A contained in the parameter space $\mathcal{A}(a,C,\epsilon)$ defined on p. 1804 in MOPM.

Assumption 3 (modified)**.** *For any $C > 0$ and $\epsilon \in (0,1)$,*

$$\lim_{K\to\infty}\lim_{T\to\infty}\inf_{A\in\mathcal{A}(0,C,\epsilon)} P_A\left(\lambda_{\min}\left(G(A,T,\epsilon)^{-1}\left[\frac{1}{T}\sum_{t=1}^{T} X_t X_t'\right]G(A,T,\epsilon)^{-1\prime}\right) \geq 1/K\right) = 1.$$

This modified assumption is identical to the old one, except for the definition of the matrix $G(A,T,\epsilon)$. Define the quasi-differenced process $\tilde y_t(A,\epsilon) \equiv (\tilde y_{1,t}(A,\epsilon),\dots,\tilde y_{n,t}(A,\epsilon))'$ by $\tilde y_{i,t}(A,\epsilon) \equiv y_{i,t} - \tilde\rho_i(A,\epsilon)y_{i,t-1}$ for all i and t. Our modified Assumption 3 then requires the sample second moment matrix of the scaled and transformed np-dimensional process

$$G(A,T,\epsilon)^{-1}X_t = \left(\frac{y_{1,t-1}}{g(\rho_1^*(A,\epsilon),T)},\dots,\frac{y_{n,t-1}}{g(\rho_n^*(A,\epsilon),T)},\tilde y_{t-1}(A,\epsilon)',\dots,\tilde y_{t-p+1}(A,\epsilon)'\right)'$$

to be asymptotically uniformly nonsingular. Note that if $p = 1$, then only the first n elements appear in the above vector. It is standard to verify the asymptotic nonsingularity of this sample second moment matrix for stationary VAR(p) parameter sequences $A = A_T$, as well as for parameter sequences that have a single local-to-unity or unit root per series $y_{i,t}$, $i = 1,\dots,n$, as assumed in the parameter space in Definition 1 of MOPM (p. 1804).[2]

[1]Since $|\tilde\rho_i(A,\epsilon)| \geq \epsilon$, the first matrix in the above display is nonsingular.

[2]Note that $g(\rho_i^*(A_T,\epsilon),T)^{-1} \propto T^{-1/2}$ for local-to-unity or unit root sequences $\rho_i(A_T)$.

See for example Hamilton (1994, pp. 551–552) for the unit root case and Stock (1994, pp. 2754–2755) for the local-to-unity case. Verifying *uniform* nonsingularity requires additional steps, as in Appendix C of MOPM.

Modified proofs

The only propositions or lemmas in MOPM (and the Supplemental Material) that rely on the specific definition of the matrix $G(A, h, \epsilon)$ are Proposition 1 and Lemmas A.3, A.5, E.1, and E.2.[3] The proofs of these results go through unchanged, except that we must additionally show that the following three statements hold for any $j \in \{1, \dots, n\}$ and $r \in \{1, \dots, p-1\}$, under the assumptions of Lemma A.3 in MOPM:

i) $\frac{1}{(T-h_T)^{1/2}} \sum_{t=1}^{T-h_T} u_t \tilde{y}_{j,t-r}(A_T, \epsilon) = O_{P_{A_T}}(1)$.

ii) $\frac{1}{(T-h_T)v(A_T,h_T,w)} \sum_{t=1}^{T-h_T} \xi_{1,t}(A_T, h_T) \tilde{y}_{j,t-r}(A_T, \epsilon) = o_{P_{A_T}}(1)$.

iii) $\frac{1}{T-h_T} \sum_{t=1}^{T-h_T} (\tilde{y}_{j,t-r}(A_T, \epsilon))^4 = O_{P_{A_T}}(1)$.

Note that if $|\rho_j(A_T)| \geq \epsilon$, then by Definition 1 in MOPM (p. 1804), $\tilde{y}_{j,t}(A_T, \epsilon) = y_{j,t} - \rho_j(A_T) y_{j,t-1}$ can be viewed as a component of a VAR($p-1$) process with coefficients $\tilde{A}_T$ contained in the uniformly stationary parameter space $\mathcal{A}(1, \epsilon, C)$. The proof of Lemma E.8 in MOPM (Supplemental Material pp. 14–15) then immediately implies that the expression on the left-hand side of (i) has uniformly bounded second moment (simply substitute $\tilde{y}_{j,t}(A_T, \epsilon)$ for $y_{j,t}$ in the proof). If on the other hand $|\rho_j(A_T)| \leq \epsilon$, then $\frac{1}{(T-h_T)^{1/2}} \sum_{t=1}^{T-h_T} u_t \tilde{y}_{j,t-r}(A_T, \epsilon) = \frac{1}{(T-h_T)^{1/2}} \sum_{t=1}^{T-h_T} u_t y_{j,t-r} \pm \epsilon \frac{1}{(T-h_T)^{1/2}} \sum_{t=1}^{T-h_T} u_t y_{j,t-r-1}$, and the proof of Lemma E.8 in MOPM shows that both these terms have uniformly bounded second moments. Statement (i) above follows.

Similarly, statement (iii) above follows directly from Markov's inequality and Lemma E.3 in MOPM (Supplemental Material p. 8). Again, in the case $|\rho_j(A_T)| \leq \epsilon$, we use convexity to derive the bound $(\tilde{y}_{j,t-r}(A_T, \epsilon))^4 \leq 8(y_{j,t-r}^4 + \epsilon^4 y_{j,t-r-1}^4)$ and treat the two terms separately.

Finally, to show statement (ii) above, it suffices by Chebyshev's inequality to show that the variance of the left-hand side is uniformly $o(1)$. For the case, $|\rho_j(A_T)| \leq \epsilon$ we can write $\frac{1}{(T-h_T)v(A_T,h_T,w)} \sum_{t=1}^{T-h_T} \xi_{1,t}(A_T, h_T) \tilde{y}_{j,t-r}(A_T, \epsilon) = \frac{1}{(T-h_T)v(A_T,h_T,w)} \sum_{t=1}^{T-h_T} \xi_{1,t}(A_T, h_T) y_{j,t-r} \pm \epsilon \frac{1}{(T-h_T)v(A_T,h_T,w)} \sum_{t=1}^{T-h_T} \xi_{1,t}(A_T, h_T) y_{j,t-r-1}$ and directly apply the proof of Lemma A.4 in

[3] See specifically the proof steps on p. 1812 and Supplemental Material pp. 7–8, 13–14, and 19. Note that certain expressions involving $G(A, h, \epsilon)$ should be transposed, as should be clear from context.

MOPM (Supplemental Material pp. 15–18) to the two terms separately. For the case $|\rho_j(A_T)| \geq \epsilon$, the variance of the left-hand side in (ii) is given by

$$\frac{1}{(T-h_T)^2}\sum_{t=1}^{T-h_T}\sum_{s=1}^{T-h_T}\frac{E[\xi_{1,t}(A_T,h_T)\tilde{y}_{j,t-r}(A_T,\epsilon)\xi_{1,s}(A_T,h_T)\tilde{y}_{j,s-r}(A_T,\epsilon)]}{v(A_T,h_T,w)^2}.$$

We now argue that the summands in the double sum are bounded by a constant times $(1-\epsilon)^{|t-s|}$, which yields the desired conclusion. Consider the case $t \geq s$ (the case $t < s$ follows by symmetry). As argued above, we can write $\tilde{y}_{j,t}(A_T,\epsilon) = \sum_{\ell=1}^{t}\beta_j(\tilde{A}_T, t-\ell)'u_\ell$, where $\|\beta_j(\tilde{A}_T,\ell)\| \leq C(1-\epsilon)^\ell$ (recall that initial conditions are zero). Assumption 1 in MOPM (p. 1793) implies

$$\begin{aligned}
&E[\tilde{y}_{j,s-r}(A_T,\epsilon)\tilde{y}_{j,t-r}(A_T,\epsilon) \mid u_{s+1},u_{s+2},\dots] \\
&= E\left[\tilde{y}_{j,s-r}(A_T,\epsilon)\sum_{\ell=1}^{s}\beta_j(\tilde{A}_T,t-r-\ell)'u_\ell \,\middle|\, u_{s+1},u_{s+2},\dots\right],
\end{aligned}$$

where we define $\beta_j(\tilde{A}_T,\ell) = 0_{n\times 1}$ for all $\ell < 0$. Thus,

$$\begin{aligned}
&|E[\xi_{1,t}(A_T,h_T)\tilde{y}_{j,t-r}(A_T,\epsilon)\xi_{1,s}(A_T,h_T)\tilde{y}_{j,s-r}(A_T,\epsilon)]| \\
&= |E\left[\xi_{1,t}(A_T,h_T)\xi_{1,s}(A_T,h_T)E\{\tilde{y}_{j,t-r}(A_T,\epsilon)\tilde{y}_{j,s-r}(A_T,\epsilon) \mid u_{s+1},u_{s+2},\dots\}\right]| \\
&= \left|E\left[\xi_{1,t}(A_T,h_T)\xi_{1,s}(A_T,h_T)\tilde{y}_{j,s-r}(A_T,\epsilon)\sum_{\ell=1}^{s}\beta_j(\tilde{A}_T,t-r-\ell)'u_\ell\right]\right| \\
&\leq \sum_{\ell=1}^{s}\|\beta_j(\tilde{A}_T,t-r-\ell)\|E\left[|\xi_{1,t}(A_T,h_T)\xi_{1,s}(A_T,h_T)\tilde{y}_{j,s-r}(A_T,\epsilon)|\,\|u_\ell\|\right] \\
&\leq \sum_{\ell=1}^{s}\|\beta_j(\tilde{A}_T,t-r-\ell)\| \times \max_{\tau_1,\tau_2,\tau_3\leq T}\left(E[\xi_{1,\tau_1}(A_T,h_T)^4]^2 \times E[\tilde{y}_{j,\tau_2}(A_T,\epsilon)^4] \times E[\|u_{\tau_3}\|^4]\right)^{1/4}.
\end{aligned}$$

Using Assumption 2, Lemma A.7, and Lemma E.3 in MOPM (pp. 1805 and 1815, and Supplemental Material p. 8), we therefore have

$$\begin{aligned}
&\frac{|E[\xi_{1,t}(A_T,h_T)\tilde{y}_{j,t-r}(A_T,\epsilon)\xi_{1,s}(A_T,h_T)\tilde{y}_{j,s-r}(A_T,\epsilon)]|}{v(A_T,h_T,w)^2} \leq \text{constant} \times \left(\sum_{\ell=1}^{s}\|\beta_j(\tilde{A}_T,t-r-\ell)\|\right) \\
&\leq \text{constant} \times \left(\sum_{\ell=1}^{s}(1-\epsilon)^{t-r-\ell}\right) \leq \text{constant} \times (1-\epsilon)^{t-s},
\end{aligned}$$

where the constants do not depend on j, r, s, t, h_T, or A_T. □

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